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Research Article

Quasilinear Stochastic Cauchy Problem in Abstract Colombeau Spaces

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Generalized solutions to the abstract Cauchy problem for a quasilinear equation with the generator of an integrated semigroup and with terms reflecting nonlinear perturbations and white noise type perturbations are under consideration. An abstract stochastic Colombeau algebra is constructed, and solutions in the algebra are studied.

1. Introduction

The paper is devoted to construction of solutions to the abstract quasilinear Cauchy problem

$$X'(t) = AX(t) + F(X) + BW(t), \quad t \geq 0, \qquad X(0) = f, \tag{1.1}$$

where A is the generator of a C_0 semigroup or an integrated semigroup in a Hilbert space H, F is a nonlinear mapping from H to H, B is a linear bounded operator from a Hilbert space $\mathbb{H}$ to H, and $W = \{W(t), t \geq 0\}$ is a stochastic process of white noise type with values in $\mathbb{H}$: $W(t) = W(t,\omega)$, $\omega \in (\Omega, \mathcal{B}(\Omega), \mu)$.

Irregularity of white noise caused by independence of random variables $W(t_1)$ and $W(t_2)$ for $t_1 \neq t_2$ and by infinite variance necessitates to define the white noise in such a way that the problem (1.1) makes certain sense.

One of the well-known ways to do this is to consider the corresponding integral equation replacing the white noise term by the integral with respect to a Wiener process W (Ito integral), written as usually in Ito theory in the following form of differentials:

$$dX(t) = AX(t)dt + F(X)dt + BdW(t), \quad t \geq 0, \qquad X(0) = f. \tag{1.2}$$

For the Cauchy problem (1.2) with generators of semigroups of class C_0, with Lipschitz nonlinearities under some growth conditions mild solutions are constructed (see, i.e., [1, 2]). In this approach questions of whether the solutions obtained are differentiable, whether they satisfy the problem (1.1), and whether the techniques can be applied for A generating regularized semigroups remain open.

Another approach, which we are going to use and generalize, is to consider (1.1) in spaces of abstract distributions, but here, due to nonlinearity of F in the equation, the problem of distribution products arises. A novel approach is to define an abstract stochastic Colombeau algebra $\mathcal{G}(\Omega, H_a)$ (see definition in Section 1) and extend the distribution approach to the algebra.

Let H_a be an algebra in a Hilbert space H, in particular, the subspace of continuous or a finitely many times differentiable functions in $L_2(\mathbb{R})$ closed under the topology of $C^k(\mathbb{R})$, $k = 0, 1, \ldots$. We consider the Cauchy problem (1.1) in the abstract stochastic Colombeau algebra $\mathcal{G}(\Omega, H_a)$ supposing that F is an infinitely differentiable mapping, $B \in \mathcal{L}(\mathbb{H}, H_a)$, and $H_a \subset \operatorname{dom} A$. We define white noise W as an element in $\mathfrak{D}'_+(\mathbb{H})$, the space of abstract distributions with values in $\mathbb{H}$ and supports in $[0, \infty)$, then by convolution with functions from $\mathfrak{D}$ we transform $W \in \mathfrak{D}'_+(\mathbb{H})$ to infinitely differentiable with respect to t functions and as a result we obtain an element BW belonging to the algebra $\mathcal{G}(\Omega, H_a)$.

As examples of A satisfying the conditions and generating different integrated semigroups one can take many of differential operators $A = P(i(\partial/\partial x))$ of correct in the sense of Petrovskiy systems [3]. The operators may be disturbed by bounded ones of any nature. For more examples see in [4, 5].

In the paper we combine the multiplication theory in Colombeau algebras which has found applications in solving differential equations, mainly hyperbolic ones (see, e.g., [6, 7]) with the theory of regularized semigroups and the theory of stochastic processes in spaces of abstract distributions (see, e.g., [8, 9]). This makes possible to solve nonlinear abstract stochastic equations with different types of white noise.

2. Definition of Abstract Colombeau Algebras

At the beginning we introduce Colombeau space of abstract (Hilbert space valued) generalized functions. For each $q \in \mathbb{N}_0$ let $\mathcal{A}_q$ be the set of all $\varphi \in \mathfrak{D}$ such that

$$\int_{\mathbb{R}} \varphi(t)dt = 1, \qquad \int_{\mathbb{R}} t^k \varphi(t)dt = 0, \quad k = 1, \ldots, q. \tag{2.1}$$

For algebra H_a in a Hilbert space H we define the space of functions $u(\varphi) := u(\varphi(t), t)$, $\varphi \in \mathcal{A}_0$, $t \in \mathbb{R}$ as follows:

$$\mathcal{E}(H_a) := (C^\infty(\mathbb{R}; H_a))^{\mathcal{A}_0} = \{u : \mathcal{A}_0 \longrightarrow C^\infty(\mathbb{R}; H_a)\}. \tag{2.2}$$

According to the definition $u(\varphi)$ is an infinitely differentiable H_a-valued function of real argument $t \in \mathbb{R}$ for each $\varphi \in \mathcal{A}_0$. Thus u can be considered as a function of two variables: $\varphi \in \mathcal{A}_0$ and $t \in \mathbb{R}$, that is,

$$u : \mathcal{A}_0 \times \mathbb{R} \longrightarrow H_a : u = u(\varphi, t), \quad \varphi \in \mathcal{A}_0,\ t \in \mathbb{R}. \tag{2.3}$$

and it is infinitely differentiable with respect to the second variable.

Differentiation and multiplication are defined as follows:

$$\begin{aligned} (uv)(\varphi) &:= u(\varphi)v(\varphi), \\ u^{(n)}(\varphi) &:= \frac{d^n}{dt^n}u(\varphi), \end{aligned} \quad \varphi \in \mathcal{A}_0. \tag{2.4}$$

The space of the H_a-valued distributions $\mathfrak{D}'(H_a)$ being a subset of the abstract distributions space $\mathfrak{D}'(H)$ is embedded in $\mathcal{E}(H_a)$ by the following mapping:

$$i : \mathfrak{D}'(H_a) \longrightarrow \mathcal{E}(H_a), \quad (iw)(\varphi) := w * \varphi, \quad w \in \mathfrak{D}'(H_a),\ \varphi \in \mathcal{A}_0. \tag{2.5}$$

Now let us introduce the functions $\varphi_\varepsilon(t) := (1/\varepsilon)\varphi(t/\varepsilon)$, $t \in \mathbb{R}$, $\varepsilon > 0$, $\varphi \in \mathcal{A}_0$ and define the linear manyfold of moderate elements $\mathcal{E}_M(H_a)$ consisting of all $u \in \mathcal{E}(H_a)$ satisfying the following condition:

(M) for each compact $K \subset \mathbb{R}$ and each $n \in \mathbb{N}_0$ there exists $q \in \mathbb{N}$ such that

$$\sup_{t\in K}\left\| \frac{d^n}{dt^n}u(\varphi_\varepsilon, t)\right\|_H = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{-q}) \quad \text{for each } \varphi \in \mathcal{A}_q. \tag{2.6}$$

To complete the definition of the Colombeau algebra of abstract generalized functions we introduce $\mathcal{N}(H_a)$ consisting of all elements $u \in \mathcal{E}(H_a)$ that satisfy the following condition:

(N) for each compact $K \subset \mathbb{R}$ and each $n \in \mathbb{N}_0$ there exists $p \in \mathbb{N}$ such that

$$\sup_{t\in K}\left\| \frac{d^n}{dt^n}u(\varphi_\varepsilon, t)\right\|_H = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{q-p}) \quad \text{for each } \varphi \in \mathcal{A}_q,\ q \geq p. \tag{2.7}$$

Elements of the space $\mathcal{E}_M(H_a)$ form a differential algebra, and $\mathcal{N}(H_a)$ is the differential ideal in it. Now define the factor algebra

$$\mathcal{G}(H_a) := \frac{\mathcal{E}_M(H_a)}{\mathcal{N}(H_a)}. \tag{2.8}$$

Similarly to $\mathcal{G}(\mathbb{R})$ (see, e.g., [6]), the algebra $\mathcal{G}(H_a)$ is an associative and commutative H_a-valued differential one. Elements of $\mathcal{G}(H_a)$ are classes of mappings. We denote them by capitals $U, V, \ldots$ and denote representatives of class $U \in \mathcal{G}(H_a)$ by corresponding small letter u.

Due to the structure theorems for abstract distributions [10], similarly to the $\mathbb{R}$-valued case, we obtain that i maps the elements of $\mathfrak{D}'(H_a)$ into $\mathcal{E}_M(H_a)$ and $i^{-1}(\mathcal{N}(H_a))$ consists of the null element of $\mathfrak{D}'(H_a)$. Thus, each element of $\mathfrak{D}'(H_a)$ is imbedded in the corresponding class of $\mathcal{G}(H_a)$ by the mapping i.

Support of an element $U \in \mathcal{G}(H_a)$ is defined as follows. We say that U is equal to zero on an open set $\Lambda \subset \mathbb{R}$ if its restriction to $\mathcal{G}_\Lambda(H_a)$ is equal to zero in $\mathcal{G}_\Lambda(H_a)$ (where algebra $\mathcal{G}_\Lambda(H_a)$ is defined in the same way as $\mathcal{G}_\Lambda(\mathbb{R}^n)$ [6]). If $w \in \mathfrak{D}'(H_a)$, then, similarly to the case of $\mathfrak{D}'(\mathbb{R})$, support of $iw \in \mathcal{G}(H_a)$ coincides with that of $w \in \mathfrak{D}'(H_a)$.

Now define $\mathcal{G}(\Omega, H_a)$, the algebra of $\mathcal{G}(H_a)$-valued random variables $\{U = U(\omega),\ \omega \in (\Omega, \mathcal{B}(\Omega), \mu)\}$ as a mapping from $(\Omega, \mathcal{B}(\Omega), \mu)$ to $\mathcal{G}(H_a)$ measurable in the following sense: there exists a representative $u \in U$ such that for any $\varphi \in \mathcal{A}_0 u^{-1}(\varphi, \cdot)$ maps any Borel subset of $\mathcal{B}(C^\infty(H_a))$ onto an element of $\mathcal{B}(\Omega)$, where the Borel σ-algebra $\mathcal{B}(C^\infty(H_a))$ on the space $C^\infty(H_a)$ is generated by the system of neighborhoods in $C^\infty(H_a)$ defined by the system of seminorms $p_{n,k}(f) = \sup_{t\in[-n,n]} \|f^{(k)}(t)\|_H$.

To complete the setting of the problem we define the generalized white noise process $W = W(\cdot, \omega)$ for each $\omega \in \Omega$, as an element in $\mathfrak{D}'_+(\mathbb{H})$, and the space of abstract distributions with values in $\mathbb{H}$ and supports in $[0, \infty)$, and then transform it into an element of $\mathcal{G}(\Omega, H_a)$.

One way to do this is based on the ideas of abstract stochastic distributions (see, e.g., [8, 9]). Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ be the space of rapidly decreasing test functions. Denote by $\mathcal{S}'(\mathbb{H})$ the space of $\mathbb{H}$-valued distributions over $\mathcal{S}$ and consider a Borel σ-algebra $\mathcal{B}(\Omega)$ generated by the weak topology of $\Omega := \mathcal{S}'(\mathbb{H})$. Then by the generalization of the Bochner–Minlos theorem to the case of Hilbert space valued generalized functions [11], there exist a unique probability measure μ on $\mathcal{B}(\Omega)$ and a trace class operator Q satisfying the condition as follows

$$\int_\Omega e^{i(\langle \omega, \theta\rangle, h)_{\mathbb{H}}} d\mu(\omega) = e^{-(1/2)\|\theta\|^2 (Qh, h)}, \quad \theta \in \mathcal{S},\ h \in \mathbb{H}. \tag{2.9}$$

(Here and below, if it is not pointed out especially, $\|\cdot\|$ denotes the norm of $L_2(\mathbb{R})$).

It makes possible to define the white noise process on $(\Omega, \mathcal{B}(\Omega), \mu)$ with values in $\mathcal{S}'(\mathbb{H}) \subset \mathfrak{D}'(\mathbb{H})$ by the identical mapping as follows:

$$\langle W(\cdot, \omega), \theta(\cdot)\rangle := \langle \omega, \theta\rangle, \quad \theta \in \mathcal{S}. \tag{2.10}$$

The above-defined process is the generalization of the corresponding real-valued Gaussian process [12], and it has zero mean and $\operatorname{Cov}\langle W, \theta\rangle = \|\theta\|^2 Q$. Define W_+ with support in $[0, \infty)$ as follows:

$$\langle W_+(\cdot, \omega), \theta(\cdot)\rangle := (-1)^k \left\langle \left(\chi \cdot W^{(-k)}\right)(\cdot), \theta^{(k)}(\cdot)\right\rangle, \quad \theta \in \mathcal{S}. \tag{2.11}$$

Here $W^{(-k)}$ is a continuous function that according to the structure theorem is a primitive of W of an order k and χ is the Heaviside function.

Another way to define a generalized $\mathbb{H}$-valued white noise on a $(\Omega, \mathcal{B}(\Omega), \mu)$, more precisely Q-white noise, is via derivative of $\mathbb{H}$-valued Q-Wiener process $\{W_Q(t), t \geq 0\}$ [9] continued by zero on $(-\infty, 0)$ as follows:

$$\langle W_+(\cdot, \omega), \theta\rangle := -\langle W_Q(\cdot, \omega), \theta'\rangle, \quad \theta \in \mathfrak{D}. \tag{2.12}$$

Finally, we map the defined generalized white noise process into the Colombeau algebra $\mathcal{G}(\Omega, H_a)$ in the following manner. By convolution with a function from $\mathcal{A}_0$ we transform $W_+(\cdot,\omega) \in \mathfrak{D}'(\mathbb{H})$ into the infinitely differential with respect to $t \in \mathbb{R}$ and measurable with respect to $\omega \in \Omega$ function w as follows:

$$w(\varphi,t,\omega) := \langle W_+(\cdot,\omega), \varphi(t-\cdot)\rangle, \quad \varphi \in \mathcal{A}_0,\ t \in \mathbb{R}. \tag{2.13}$$

So, $w(\varphi,\cdot,\omega) \in C^\infty(\mathbb{R},\mathbb{H})$, $\varphi \in \mathcal{A}_0$, $\omega \in \Omega$ a.s., and $w(\varphi,\cdot,\cdot) \in C^\infty(\mathbb{R}, L_2(\Omega,\mathbb{H}))$.

Let $B \in \mathcal{L}(\mathbb{H}, H_a)$, and then applying B to w we obtain that $Bw(\varphi,t,\omega) \in H_a$ and the map

$$Bw(\varphi,\cdot,\cdot) : \mathcal{A}_0 \longrightarrow C^\infty(\mathbb{R}, L_2(\Omega, H_a)) \tag{2.14}$$

are representative of a class in $\mathcal{G}(\Omega, H_a)$. The corresponding class we denote by BW. Since the support of $W_+ \in \mathfrak{D}'(\mathbb{H})$ is $[0;\infty)$, by definition of support of an element of $\mathcal{G}(\Omega, H_a)$ we have $\operatorname{supp} BW = [0;\infty)$. That is the sense we attach to the stochastic term in (1.1).

3. Solutions to the Cauchy Problem with Infinitely Differentiable Nonlinearities

Let $H = \mathbb{H} = L_2(\mathbb{R})$ and the domain of A lie in the set of continuous functions of $L_2(\mathbb{R})$. Let H_a be the set of finitely many times differentiable functions of $L_2(\mathbb{R})$ and $H_a \subseteq \operatorname{dom} A$. Then multiplication of elements of $L_2(\mathbb{R})$ is well defined on the set $L_2(\mathbb{R}) \cap \operatorname{dom} A$ as pointwise continuous functions multiplication.

In this section for the problem (1.1), where nonlinearity F is infinitely differentiable, bounded with all its derivatives, and has the property $F(0) = 0$ and where stochastic term BW is constructed above, we will search a solution as an element of the abstract stochastic Colombeau algebra $\mathcal{G}(\Omega, H_a)$. Since H_a is chosen as the set of finitely many times differentiable functions in $L_2(\mathbb{R})$, operator $B \in \mathcal{L}(\mathbb{H}, H_a)$ can be taken, for example, as convolution with a finitely many times differentiable function from $L_2(\mathbb{R})$ and with condition of the convolution existence.

Suppose at the beginning that A generates a C_0-semigroup $\{V(t), t \geq 0\}$ in $L_2(\mathbb{R})$.

Consider the question whether there is a solution to the problem

$$Y' = AY + F(Y) + BW, \quad \operatorname{supp} Y \subseteq [0,\infty), \tag{3.1}$$

as an element of algebra $\mathcal{G}(\Omega, H_a)$. To do this, for an arbitrary $\eta > 0$, we consider $Bw(t) := Bw(\varphi,t,\omega)$, $\varphi \in \mathcal{A}_0$, $\omega \in \Omega$, with support in $[-\eta,\infty)$ as a representative of the white noise term $BW \in \mathcal{G}(\Omega, H_a)$ [7]. By definition of elements of $\mathcal{G}(\Omega, H_a)$, for each fixed $\varphi \in \mathcal{A}_0$, $Bw(t)$ is an infinitely differentiable function of $t \in \mathbb{R}$ with values in H_a and measurable with respect to $\omega \in \Omega$. Let us take an arbitrary $\varphi \in \mathcal{A}_0$ and consider the problem

$$y'(t) = Ay(t) + F\big(y(t)\big) + Bw(t), \quad t \geq -\eta, \qquad y(t) = 0, \quad t \leq -\eta, \tag{3.2}$$

where $y(t) = y(\varphi,t,\omega)$, $\varphi \in \mathcal{A}_0$, $\omega \in \Omega$. We will search a solution of this problem belonging to $C^\infty([-\eta;\infty); \operatorname{dom} A)$ for ω a.s.

Consider the equation

$$y(t) = \int_{-\eta}^{t} V(t-s)F\big(y(s)\big)ds + \int_{-\eta}^{t} V(t-s)Bw(s)ds =: Qy(t), \quad t \geq -\eta. \tag{3.3}$$

The introduced operator Q is a Volterra type one. Using the differentiability of F and boundedness of its derivative let us show that Q^k (where $k = k(T)$) is a contraction on the segment $[-\eta; T]$.

Since F is differentiable, we have $F(\mu) - F(\lambda) = F'(\xi)(\mu - \lambda)$, $\xi \in (\mu; \lambda)$, for any $\mu, \lambda \in \mathbb{R}$. Then for any $y(\cdot)$ and $z(\cdot)$ we get the pointwise equality

$$F\big(y(s)\big) - F(z(s)) = F'(\xi)\big(y(s) - z(s)\big), \quad s \in [-\eta; \infty), \tag{3.4}$$

where ξ is an appropriate point from $(y(s); z(s))$ and the following estimate holds:

$$\big\|F\big(y(s)\big) - F(z(s))\big\| \leq L\big\|y(s) - z(s)\big\|, \qquad L = \max_{\xi \in \mathbb{R}}\big|F'(\xi)\big|. \tag{3.5}$$

This and exponential boundedness of C_0 semigroups:

$$\|V(t)\|_{\mathcal{L}(L_2(\mathbb{R}))} \leq Ce^{at}, \tag{3.6}$$

for each $t \in [-\eta; \infty)$ imply that

$$\big\|Qy(t) - Qz(t)\big\| \leq CLe^{a(t+\eta)}(t+\eta)\max_{s \in [-\eta; t]}\big\|y(s) - z(s)\big\|. \tag{3.7}$$

For squares we have

$$\begin{aligned} Q^2y(t) - Q^2z(t) &= \int_{-\eta}^{t} V(t-s)F\big(Q\big(y(s)\big)\big)ds - \int_{-\eta}^{t} V(t-s)F(Q(z(s)))ds \\ &= \int_{-\eta}^{t} V(t-s)\big[F\big(Q\big(y(s)\big)\big) - F(Q(z(s)))\big]ds. \end{aligned} \tag{3.8}$$

Then we have

$$\Big\|Q^2y(t) - Q^2z(t)\Big\| \leq C^2L^2e^{2a(t+\eta)}\frac{(t+\eta)^2}{2}\max_{s \in [-\eta; t]}\big\|y(s) - z(s)\big\|, \tag{3.9}$$

and for every $k \in \mathbb{N}$

$$\Big\|Q^ky(t) - Q^kz(t)\Big\| \leq C^kL^ke^{ka(t+\eta)}\frac{(t+\eta)^k}{k!}\max_{s \in [-\eta; t]}\big\|y(s) - z(s)\big\|, \tag{3.10}$$

hence

$$\max_{t\in[-\eta;T]}\left\|Q^k y(t) - Q^k z(t)\right\| \le C^k L^k e^{ka(T+\eta)} \frac{(T+\eta)^k}{k!} \max_{t\in[-\eta;T]}\|y(t) - z(t)\|. \tag{3.11}$$

The constant in this estimate can be made less then unity by choosing $k = k(T)$. Thus Q^k is the contraction, and the sequence of approximations $y_n(t) = Q^{nk} y_0(t)$ has the limit in H:

$$y(t) = \lim_{n\to\infty} Q^{nk} y_0(t), \tag{3.12}$$

uniform with respect to $t \in [-\eta; T]$.

Note that if one takes an infinitely differentiable with respect to t function $y_0(\cdot)$ as the first point for the approximating sequence, then function

$$z(t) = Qy_0(t) = \int_{-\eta}^{t} V(t-s)F(y_0(s))ds + \int_{-\eta}^{t} V(t-s)Bw(s)ds, \quad t \ge -\eta, \tag{3.13}$$

is also an infinitely differentiable with respect to t function, and consequently $y_1(\cdot) = Q^k y_0(\cdot)$ has the same property as well as all subsequent iterations $y_n(\cdot)$.

It can be shown by the same arguments that the sequence $y_n'(\cdot)$ converges to its limit in H uniformly with respect to $t \in [-\eta; T]$; hence $y(\cdot)$ is differentiable and $y'(t) = \lim_{n\to\infty} y_n'(t)$. Similarly it can be shown that $y(\cdot)$ is infinitely differentiable function with values in H.

Now we show that $y_n(t) \in H_a$ if $y_0(t) \in H_a$, $t \ge 0$. Let $t \ge 0$ be fixed. Note firstly that $F(\alpha) = \mathcal{O}(\alpha)$ as $\alpha \to 0$. Really, due to the infinite differentiability of F and property $F(0) = 0$ $F(\alpha)$ can be repersented by the Taylor series with first term proportional to α. Then, since $y_0(t) \in H_a$, it is differentiable with respect to variable of $L_2(\mathbb{R})$, and $y_0(t) \to 0$ as variable of $L_2(\mathbb{R})$ tends to infinity. Thus, $F(y_0(t)) = \mathcal{O}(y_0(t))$ as variable of $L_2(\mathbb{R})$ tends to infinity.

Further, semigroup operators $V(t)$ map $L_2(\mathbb{R})$ into $L_2(\mathbb{R})$, and, moreover, they map differentiable functions (with respect to variable of $L_2(\mathbb{R})$) into the set of differentiable ones due to their boundness. It follows that

$$V(t-s)F(y_0(s)) : H_a \longrightarrow H_a, \qquad V(t-s)Bw(s) \in H_a, \tag{3.14}$$

hence Q acts in H_a and $y_1(t) = Q^k y_0(t) \in H_a$ as well as $y_n(t)$ for any $n \in \mathbb{N}$.

Thus, we obtain $y_n(t) \in H_a$, but in the general case $\lim_{n\to\infty} y_n(t) = y(t)$ does not belong to H_a since algebra H_a is not closed in the sense of $L_2(\mathbb{R})$ convergence.

If $y(t) \in H_a$, then we show that it is a representative of a class from $\mathcal{G}(\Omega, H_a)$. As it is known (see, e.g., [4]), if $\mathcal{F}(\cdot)$ is a differentiable function or $\mathcal{F}(t) \in \operatorname{dom} A$ for any $t \ge 0$, then solution of the inhomogeneous abstract Cauchy problem

$$u'(t) = Au(t) + \mathcal{F}(t), \quad t \ge 0, \qquad u(0) = 0, \tag{3.15}$$

with A generating a C_0 semigroup of operators $\{V(t), t \ge 0\}$ exists and is defined by the formulae $u(t) = \int_0^t V(t-s)\mathcal{F}(s)ds$. Since in the case under consideration $F(y(t))$ as well as any representative $Bw(t)$ of white noise process are $L_2(\mathbb{R})$-valued infinitely differentiable with respect to t functions, the solution of (3.3) is a solution to the problem (3.2).

Due to the property of C_0 semigroup $V(t) = 0$ as $t < 0$, it follows from (3.3) that $y(t) = 0$ as $t \leq -\eta$; that is, support of the obtained solution lies in $[-\eta; \infty)$.

Now we show that y is a representative of a class $Y \in \mathcal{G}(\Omega, H_a)$, that means that it satisfies the condition (M) almost everywhere. It follows from differentiability of F and the condition $F(0) = 0$ that for each $s \in [-\eta; t]$ the following equality

$$F(y(\varphi, s, \omega)) = F(y(\varphi, s, \omega)) - F(0) = F'(\xi)y(\varphi, s, \omega), \tag{3.16}$$

where $\xi \in (0; y(\varphi, s, \omega))$, $\varphi \in \mathcal{A}_0$, $\omega \in \Omega$, holds. Now for an arbitrary compact $K \subset \mathbb{R}$ from (3.3) we obtain that

$$\begin{aligned}\max_{t\in K}\|y(\varphi_\varepsilon, t, \omega)\| \leq C\max_{t\in K} e^{a(t+\eta)}\|Bw(\varphi_\varepsilon, t, \omega)\|\\ + C\max_{\xi\in\mathbb{R}}\|F'(\xi)\|\max_{t\in K} e^{a(t+\eta)}\int_{-\eta}^{t}\|y(\varphi_\varepsilon, s, \omega)\|ds,\end{aligned} \tag{3.17}$$

and due to boundedness of F' we have

$$\max_{t\in K}\|y(\varphi_\varepsilon, t, \omega)\| \leq C_1\max_{t\in K}\|Bw(\varphi_\varepsilon, t, \omega)\| + C_2\max_{t\in K}\int_{-\eta}^{t}\|y(\varphi_\varepsilon, s, \omega)\|ds. \tag{3.18}$$

Since Bw is a representative of class BW from $\mathcal{G}(\Omega, H_a)$, the first term in the right-hand side of the inequality for every $\varphi \in \mathcal{A}_q$ increases as $\varepsilon \to 0$ not faster than ε^{-q} for some $q \in \mathbb{N}$. Then, due to Gronwall-Bellmann inequality the left-hand side behaves in the same way, that proves the condition (M) with $n = 0$. Let us remind that the Gronwall-Bellmann inequality states that if

$$y(t) \leq c + \int_{t_0}^{t} f(s)y(s)ds, \quad c > 0,\ t > t_0, \tag{3.19}$$

for positive continuous functions y and f, than $y(t) \leq ce^{\int_{t_0}^{t} f(s)ds}$. Behavior of derivatives of $y(\cdot)$ can be checked up in the same manner using that F is infinitely differentiable and its derivatives are bounded.

Now let us show that supp $Y \subseteq [0; \infty)$. To do this we consider two solutions of (3.2) $y_{\eta_1}(\cdot)$ and $y_{\eta_2}(\cdot)$ corresponding $\eta_1 \neq \eta_2$ and verify that difference $y_{\eta_1}(\cdot) - y_{\eta_1}(\cdot)$ belongs to $\mathcal{N}(H_a)$. Note that $\eta = \max\{\eta_1, \eta_2\}$. Then we have

$$\begin{aligned}y'_{\eta_1}(t) - y'_{\eta_2}(t) = A(y_{\eta_1}(t) - y_{\eta_2}(t)) + F(y_{\eta_1}(t)) - F(y_{\eta_2}(t)) + g(t), \quad t \geq -\eta,\\ y_{\eta_1}(t) - y_{\eta_2}(t) = 0, \quad t \leq -\eta,\end{aligned} \tag{3.20}$$

where $g \in \mathcal{N}(H_a)$ as the difference of two representatives of the stochastic term BW whose support is in $[0;\infty)$. Then, similar to (3.18), we obtain the following estimation:

$$\begin{aligned}\max_{t\in K}\|y_{\eta_1}(\varphi_\varepsilon,t,\omega)-y_{\eta_2}(\varphi_\varepsilon,t,\omega)\| &\leq C_1\max_{t\in K}\|g(\varphi_\varepsilon,t,\omega)\| \\ &\quad + C_2\max_{t\in K}\int_{-\eta}^{t}\|y_{\eta_1}(\varphi_\varepsilon,s,\omega)-y_{\eta_2}(\varphi_\varepsilon,t,\omega)\|ds.\end{aligned} \tag{3.21}$$

Since the first term satisfies to (N), Gronwall-Bellman lemma implies that $y_{\eta_1}(t)-y_{\eta_2}(t) \in \mathcal{N}(H_a)$, so $\operatorname{supp} Y \subseteq [0;\infty)$.

Now we show that the solution of (3.1) is unique in the algebra $\mathcal{G}(\Omega,H_a)$. Let $Y_1,Y_2 \in \mathcal{G}(\Omega,H_a)$ with support in $[0;\infty)$ be two solutions of (3.1). Then for any representatives y_1, y_2 of these classes and each $\eta>0$ the following relations hold:

$$\begin{gathered}y_1'(t)-y_2'(t)=A\big(y_1(t)-y_2(t)\big)+F\big(y_1(t)\big)-F\big(y_2(t)\big)+g(t), \quad t\geq-\eta, \\ y_1(t)-y_2(t)\in\mathcal{N}(H_a), \quad t\leq-\eta,\end{gathered} \tag{3.22}$$

where g is an element of $\mathcal{N}(H_a)$. Then, as above, Gronwall-Bellman Lemma implies that $y_1(t)-y_2(t)\in\mathcal{N}(H_a)$, that is $Y_1-Y_2=0$ in $\mathcal{G}(\Omega,H_a)$.

Taking into account that the linear Cauchy problem, corresponding to (1.1) has the following form in spaces of distributions:

$$\langle X',\varphi\rangle=\langle\delta,\varphi\rangle f+\langle AX,\varphi\rangle+\langle BW,\varphi\rangle, \quad \varphi\in\mathfrak{D}, \tag{3.23}$$

and the required solution to the Cauchy problem (1.1) in $\mathcal{G}(\Omega,H_a)$ is related with the obtained Y as follows: $X(t)=Y(t)+(V*i\delta)(t)f$, $t\geq 0$, $f\in H_a$.

In the general case since the limit of $y_n(t) \notin H_a$, we obtain only the approximated solutions of (3.3)—the fundamental sequence $\{y_n\}$ obtained by the following equalities

$$y_n(t)=Q^k y_{n-1}(t), \quad t\leq-\eta. \tag{3.24}$$

So, we get the following result.

Theorem 3.1. *Let A be the generator of a C_0-semigroup $\{V(t),\ t\geq 0\}$ in $L_2(\mathbb{R})$. Let F be an infinitely differentiable function in $\mathbb{R}$, bounded with all its derivatives and $F(0)=0$. Let $B\in\mathcal{L}(L_2(\mathbb{R}),H_a)$ and BW be an element of $\mathcal{G}(\Omega,H_a)$ with representative Bw defined by* (2.13)*. Then for any $\eta>0$ and $\varphi\in\mathcal{A}_0$ there exists the unique solution of* (3.2) *$y\in C^\infty([-\eta;\infty);H)$. If $y\in C^\infty([-\eta;\infty);H_a)$, then* (3.1) *has the unique solution in algebra $\mathcal{G}(\Omega,H_a)$. In this case the solution to the Cauchy problem* (1.1) *in $\mathcal{G}(\Omega,H_a)$ is $X=Y+(V*i\delta)f$ for any $f\in H_a$.*

Now consider the case of A generating an integrated semigroup. If operator A generates an exponentially bounded n-times integrated semigroup $\{V_n(t), t\geq 0\}$, then solution operators $V(t)$ of homogeneous Cauchy problem are defined as follows: $\langle V, \varphi \rangle = (-1)^n \langle V_n, \varphi \rangle$, $\varphi \in \mathfrak{D}$, and instead of (3.3) we have the following equation:

$$\begin{aligned} y(\varphi, t, \omega) = &\int_{-\eta}^{t} V_n(t-s) F^{(n)}(y(\varphi, s, \omega))ds \\ &+ \int_{-\eta}^{t} V_n(t-s)(Bw)^{(n)}(\varphi, s, \omega)ds, \quad \varphi \in \mathcal{A}_0,\ \omega \in \Omega,\ t \geq -\eta. \end{aligned} \tag{3.25}$$

Here all derivatives (in t) exist due to infinite differentiability of F and Bw. Using the equality, similarly to the case of semigroups of class C_0, we obtain the corresponding approximations y_n and the solution in $\mathcal{G}(\Omega, H_a)$ if the limit of $y_n(t)$ belongs to H_a.

4. Conclusions

In conclusion we note that the present paper is only the beginning of researches of abstract stochastic equations with nonlinearities in Colombeau algebras. Among important questions that remain open and are supposed to be investigated in future are convergence of solutions $y(\varphi_\varepsilon)$ as $\varepsilon \to 0$, equations with generators of more general regularized semigroups, and equations in arbitrary Hilbert spaces.

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